A Spatial Extension to Traditional Regression-based Vertical Inequity Measures

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Introduction

For many local governments, property tax is a vital source of revenue, offering financial autonomy and enabling reinvestment into local infrastructure. Since real estate is immovable, property tax ties revenue generation to the locality, allowing governments to fund improvements that help maintain or increase property values, mitigate negative neighbourhood effects (Jacobs, 1961; Wilson & Kelling, 1982), ensure protection of residents' most valuable assets—often central to retirement strategies (Doling & Ronald, 2010; Piketty, 2014). Moreover, property taxes can incentivize efficient land use through rate differentiation and related policies (Tideman, Akobundo, Johns, & Wutthicharoen, 2002), whilst also providing essential revenue for climate change adaptation measures at the local level.

Various bases for property taxation exist, with the ad valorem model being one of the most widely used. Under this approach, tax liability is linked to a property's market value. Regardless of the base used, core principles in property taxation include explainability and willingness to pay, both of which are strengthened when assessments are perceived as fair, accurate, and consistent. Horizontal inequity refers to the differences in valuation quality between different types of real estate, such as bungalows and row houses. Vertical inequity refers to the differences in valuation quality between different price strata of real estate. Therefore, vertical inequity can be interpreted as over taxing or under taxing either the "rich" or the "poor". Consequently, the concepts of horizontal and vertical (in)equity have long dominated academic discourse (Sirmans et al., 2008; Carter, 2016; Quintos et al., 2023). Indeed, while early debates emphasized the need for identifying the best vertical equity measures, more recent times have seen discussions focusing on examining measurement limitations and employing multiple methods for analysis (Carter, 2016; McMillen & Singh, 2023; Quintos et al., 2023). Mostly, these measurement methods have focussed on global results. This paper expends the school of thought of local vertical equity measures since vertical equity could also be present in both ways on the local level, where the global level might only show one outcome. This identifies more intricate patterns that can occur from using newer, more spatially aware valuation models.

Accordingly, vertical equity measurements face three persistent challenges. First, because true market value is unobservable, these metrics often suffer from measurement error bias (Clapp, 1990). Secondly, given the wide range of variables influencing real estate values, omitting relevant factors leads to omitted variable bias (Birch, Sunderman, & Radetsky, 2017). Thirdly, data aggregation issues can distort results (McCord et al., 2022; Hermans et al., 2023). In extreme cases, this leads to Simpson's paradox, where aggregated data tells a different story than disaggregated subsets (Simpson, 1951). Consequently, it is argued that this is just an alternative way to interpret data (Ellenberg, 2022). Moreover, a spatial variant of this issue is also observable — the modifiable areal unit problem (MAUP), which arises when aggregating point data into neighbourhoods alters statistical outcomes depending on how those neighbourhoods are defined (Openshaw, 1984). This MAUP has gained particular attention in real estate assessment. Techniques such as geographically weighted regression (GWR) have been used to enhance valuation models (Brunsdon, Fotheringham & Charlton, 1998; McMillen, 1996). More recently, Bidanset et al. (2019) applied GWR to vertical equity measures, addressing the aggregation issue and overcoming MAUP, building upon the previous work into neighbourhood-level vertical equity analysis for localized approaches (Birch, Sunderman, & Radetsky, 2017).

This study therefore contributes to local regression-based vertical inequity measurement by applying the GWR framework to earlier metrics excluded in the work of Bidanset et al. (2019). These metrics are: Paglin-Fogarty (1972), Cheng (1974), IAAO (1978), Kochin-Parks (1982) and Bell (1984). In doing so, it also introduces an alternative significance testing approach. The structure is as follows: a theoretical overview of GWR and vertical equity measures; methodology; application to a tailored automated valuation model for Haarlem, The Netherlands; followed by results, discussion, conclusions, and suggestions for future research.

Theory and Literature

The method of least squares emerged in the early 19th century, with French mathematician Legendre (1805) using it to model cometary motion. Gauss (1821) later refined it for similar purposes, but the term "regression" was coined by Galton (1890), who applied it in studying heredity. Initially limited by its computational demands, the method became widespread with the rise of modern computing, and it has a foundational basis across numerous industries due to its flexibility and accessibility.

A significant advancement came in the late 1990s with the introduction of GWR. This method performs localized regressions for each observation, weighting nearby data more heavily than observations further away. In doing so, GWR incorporates Tobler's First Law of Geography: "Everything is related to everything else, but near things are more related than distant things" (Tobler, 1970; McMillen, 1996; Brunsdon, Fotheringham, & Charlton, 1998). The scope of observations used in each local regression depends on a bandwidth parameter, determined either by geographic distance or through optimization criteria like Akaike Information Criterion (AIC) or cross-validation (CV). The weighting is implemented through kernel functions, which assign weights based on distance. Common kernels include the Gaussian kernel, which uses all observations with decreasing weight by distance, and the bi-square kernel, which limits analysis to observations within a specific range. Other kernels include box-car, exponential, and tricube, with each applying a distinct function to determine how influence diminishes with distance (Table 1).

The GWR technique helps overcome the MAUP by conducting regressions at the observation level. This bypasses the biases introduced by predefined geographic boundaries and is related to broader aggregation issues. Thus, because GWR produces individualized regressions, it avoids the distortion seen in global models affected by aggregation (Borst, 2014; Brunsdon et al., 1998). This is also the core reason why extending vertical equity measures into a spatial framework can help to identify granular patters of inequity. Nonetheless, GWR also faces challenges in significance testing and can be more prone to Type I errors (false positives). However, over time multiple correction methods have been developed to mitigate these challenges,

TABLE 1 ▸ KERNAL WEIGHTING FUNCTIONS

Kernel		
Bi-square	$\omega_{ij} = (1 - (d_{ij}/r)^2)^2$ if $d_{ij} \leq r$	$\omega_{ij} = 0$ otherwise
Box-Car	$\omega_{ij} = 1$ if $d_{ij} \leq r$	$\omega_{ij} = 0$ otherwise
Exponential	$\omega_{ij} = exp(-d_{ij}/b)$	
Gaussian	$\omega_{ij} = exp(-d_{ij}^2/2b^2)$	
Tricube	$\omega_{ij} = (1 - (d_{ij}/r)^3)^3$ if $d_{ij} \leq r$	$\omega_{ij} = 0$ otherwise

including the Benjamini-Hochberg, Benjamini-Yekutieli, Bonferroni, and Fotheringham-Byrne procedures (Lu, Harris, Charlton, & Brunsdon, 2014). Considering this, this paper adopts the Bonferroni correction, which adjusts the significance level α by dividing it by the number of comparisons, offering a straightforward safeguard against overestimating significance.

Model performance in real estate valuation for property tax purposes

Model performance in real estate valuation has been a central topic for over fifty years. Performance indicators typically rely on the discrepancy between model outcomes and a market value indicator—most commonly, the assessment-to-sale price ratio, with sale price serving as the market proxy. Model performance is generally evaluated through three key concepts. Firstly, accuracy, or the model's alignment with the market, is often assessed using a central tendency measure of the assessment-to-sale price ratio. Secondly precision, the model's variability around that central tendency, commonly measured using the coefficient of dispersion (COD) in the context of property tax assessment. Thirdly is consistency, typically divided into horizontal equity, which is the model's uniformity across property groupings (e.g., by category or neighbourhood), and vertical equity, which examines how well the model performs across different price strata. Thus, vertical equity can be regressive, where lower-valued properties are over-assessed relative to higher-valued ones, or progressive, where the reverse occurs (IAAO 2013). This paper specifically focuses on regression-based vertical equity measures, namely those that were not explored in Bidanset et al. (2019)[1]. Thereby offering more tools for the current academic consensus that multiple vertical equity measures should be used before conclusions can be drawn.

Traditional regression-based vertical equity measures

The discussion on regression-based vertical equity measures began in the seminal work of Paglin and Fogarty (1972), who argued that sale price is the best indicator of market value. It should appear on the explanatory (right-hand) side of the equation, assuming a linear relationship. Cheng (1974) built on this by introducing a double-logarithmic transformation to account for potential non-linearity in the relationship. The International Association of Assessing Officers (IAAO, 1978) supported the use of sale price as the preferred market indicator but recommended explaining the assessment-to-sale price ratio discrepancy rather than the raw assessed value — an approach aligned with earlier work by Oldman and Aaron (1965). However, a contrasting view came from Kochin and Parks (1982), who accepted Cheng's non-linear transformation but argued that the assessed value, not sale price, should serve as the market value indicator — thus reversing the direction of explanation. However, this was soon challenged by Bell (1984), who reverted to using sale price as the market indicator but proposed a quadratic extension to better capture non-linear relationships (Table 2). Currently, there is now broad agreement that using sale price as the market value indicator introduces a bias toward detecting regressivity, alternatively using assessed value tends to suggest progressivity (Clapp, 1990; McMillen & Singh, 2023).

The regression framework used in the traditional vertical equity measures is presented in Equation 1, with the spatial extension method highlighted in Equation 2.

$$y_i = \beta_0 + \sum_k \beta_k \chi_{ik} + \varepsilon_i \qquad (1)$$

$$y_i = \beta_0(X_i, Y_i) + \sum_k \beta_k(X_i, Y_i)\chi_{ik} + \varepsilon_i \qquad (2)$$

Where: y_i = the i-th dependent variable; β_0 = the model intercept; β_k = the k-th coefficient; χ_{ik} = the k-th variable for the i-th dependent variable; ε_i = the error term for the i-th dependent variable; (X_i, Y_i) = the x, y coordinates of the i-th regression point.

TABLE 2 ▸ **TRADITIONAL VERTICAL EQUITY MEASURE FORMULAS AND THEIR SPATIAL EXTENSIONS**

Measure	Formula	Spatial extension
Paglin-Fogarty (1972)	$AV_i = \alpha_0 + \alpha_1 SP_i$	$AV_i = \alpha_0 (X_i, Y_i) + \alpha_1 (X_i, Y_i) SP_i$
Cheng (1974)	$lnAV_i = \alpha_0 + \alpha_1 lnSP_i$	$lnAV_i = \alpha_0 (X_i, Y_i) + \alpha_1 (X_i, Y_i) lnSP_i$
IAAO (1978)	$\frac{AV_i}{SP_i} = \alpha_0 + \alpha_1 SP_i$	$\frac{AV_i}{SP_i} = \alpha_0 (X_i, Y_i) + \alpha_1 (X_i, Y_i) lnSP_i$
Kochin-Parks (1982)	$lnSP_i = \alpha_0 + \alpha_1 lnAV_i$	$lnSP_i = \alpha_0 (X_i, Y_i) + \alpha_1 (X_i, Y_i) lnAV_i$
Bell (1984)	$AV_i = \alpha_0 + \alpha_1 SP_i + \alpha_2 SP_i^2$	$AV_i = \alpha_0 (X_i, Y_i) + \alpha_1 (X_i, Y_i) SP_i + \alpha_2 (X_i, Y_i) SP_i^2$

Note: AVi represents the i-th assessed value, SPi represents the i-th sale price, α_0 represents the intercept term. α_1 and α_2 represent the regression coefficients.

The Geographically Weighted Regression (GWR) model incorporates both a bandwidth (or optimal number of neighbours) and a kernel. The model employs the Gaussian kernel for weighting observations, with the number of neighbours optimized based on the corrected Akaike Information Criterion (AICc).

Methodology

To test the vertical equity measures and their spatial extensions, a custom automated valuation model (AVM) is created for Haarlem, The Netherlands, with a population of approximately 170,000. The model focuses on single-family residential real estate, using 3,821 sales from January 1, 2019, to December 31, 2022. This AVM is developed for research purposes and does not reflect the municipality's actual property tax assessments. The outcomes of this (global) AVM will be analysed using the observed sale prices in the municipality, using both the traditional vertical equity measures and their spatially extended counterparts.

Results

The assessment of the model performance of the exemplary AVM made for the municipality of Haarlem is undertaken in three stages. First, the model performance indicators proposed in the IAAO Standard on Ratio Studies (2013) are presented (Table 3). For brevity reasons, consistency results are only considered for vertical equity. Considering the accuracy of the AVM, the median ratio must be interpreted – with a median ratio close to 1 indicating good accuracy. The median ratio for the Haarlem AVM scores a median ratio of 1.005 in the training set and 1.018 in the test set (holdout). While the accuracy drops slightly in the test set, it is within acceptable limits. The precision of the AVM is measured through the coefficient of dispersion (COD). With a 13.49% COD in the training set and a slightly improved 13.21% in the test set, this falls within the IAAO ratio standards bandwidth (5%-15%). Both the PRD and PRB are measures for vertical equity. The 1.028 PRD results for the training set and the slightly improved 1.024 PRD result for the test set indicate a slight presence of regressive vertical inequity, but within the accepted bandwidth ranging between 0.98 to 1.03. The statistically significant -0.180 PRB result for the training set and the statistically significant -0.136 PRB result for the test set both indicate a presence of regressive vertical inequity.

TABLE 3 ▸ **GLOBAL MODEL PERFORMANCE RESULTS FOR IAAO STANDARD ON RATIO STUDIES MEASURES**

	Train	Test
Median Ratio	1.005	1.018
COD	13.49%	13.21%
PRD	1.028	1.024
PRB	-0.180*	-0.136*

*Denotes statistical significance at the $p<0.05$ level.

The results for the traditional regression-based

TABLE 2 ▸ GLOBAL RESULTS FOR TRADITIONAL REGRESSION-BASED VERTICAL EQUITY MEASURES

	Paglin Fogarty	Cheng	IAAO	Kochin Parks	Bell
Training	149132.1*	0.74*	-0.00000047*	0.96*	91386.0*
Test	139099.5*	0.80*	-0.00000043*	0.80*	7446.9*
judgement	Regressive	Regressive	Regressive	Progressive	Regressive

*Denotes statistical significance at the $p<0.05$ level.

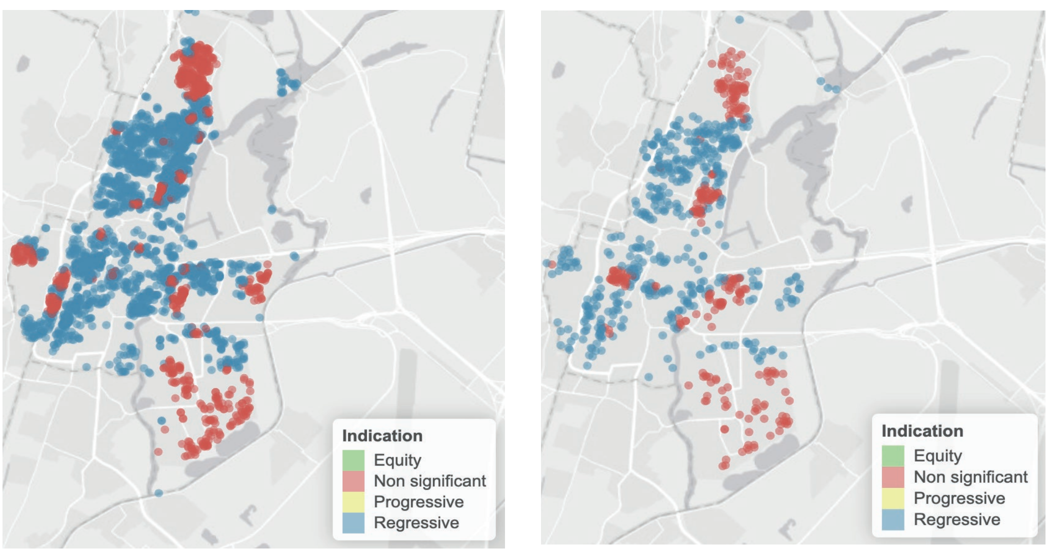
vertical equity measures are presented in Table 4. The Paglin and Fogarty test for both the training and test set of the Haarlem AVM have statistically significant variables of interest indicating a presence of regressive vertical inequity. Similarly, for the Cheng approach, both the training and the test set exhibit the presence of regressive vertical inequity. The IAAO regression model also indicates the presence of vertical inequity for both training and test sets. This is similar to findings for the Bell model, which suggests the presence of regressive vertical inequity. In contrast, the Kochin and Parks results, however, indicate the presence of progressive vertical inequity.

SPATIALLY EXTENDED MODEL RESULTS

Since the spatially extended version of the traditional regression-based vertical equity measure produces a different result for each object in the study, the findings are presented visually (geographically). The level of statistical significance is based on the Bonferroni corrected p-values at the 1% level, considering the proneness of the GWR framework to introduce Type I errors.

Figure 1 presents the training set results for the spatially extended Paglin-Fogarty vertical equity measure. The majority of the regression observations exhibit the presence of regressive vertical inequity, and a spatial pattern can be seen between the training and test set. 1a;1b). Both the north and south of the municipality of Haarlem show non-significant returns for the spatially extended vertical equity measure. Further, both

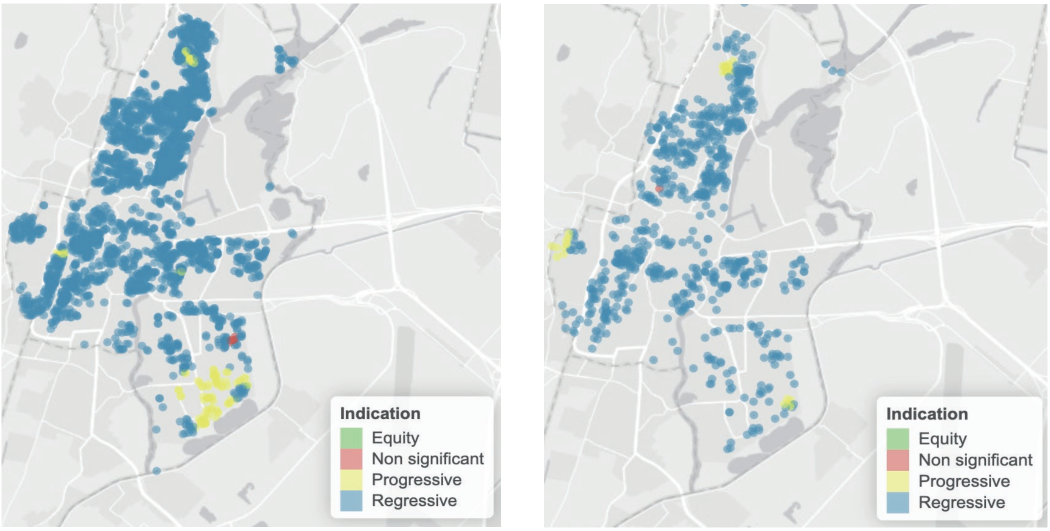
FIGURE 1 ▸ SPATIALLY EXTENDED PAGLIN-FOGARTY MEASURE RESULTSS



a) Training

b) Test

FIGURE 2 ▸ SPATIALLY EXTENDED PAGLIN-FOGARTY MEASURE RESULTS



the training and holdout (test) findings exhibit pockets of non-significance throughout the municipality.

For the spatially extended Cheng vertical equity measure, the results are different to the extended Paglin-Fogarty measure. The findings show a dominance of vertical equity in the form of regressivity. However, the southern and northern districts, which had a non-significant outcome in the spatially extended Paglin-Fogarty measure, exhibit progressivity in terms of vertical equity for both the training and test datasets. In addition, the number of non-significant results decreased in comparison to the extended Paglin-Fogarty measure.

As observed in Figure 3, the findings for both the training and test spatially extended IAAO vertical equity measure show similar spatial trends and

FIGURE 3 ▸ SPATIALLY EXTENDED PAGLIN-FOGARTY MEASURE RESULTS

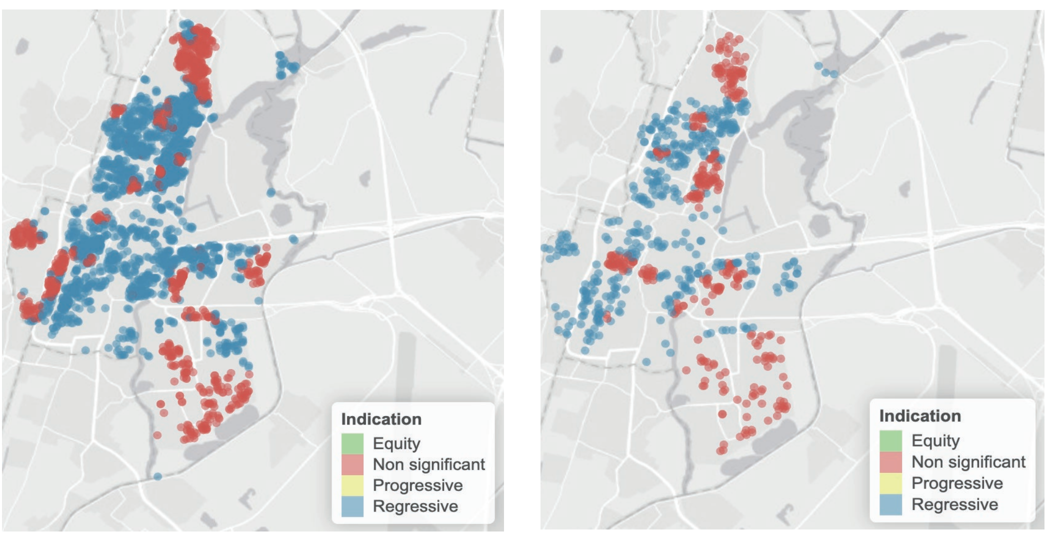
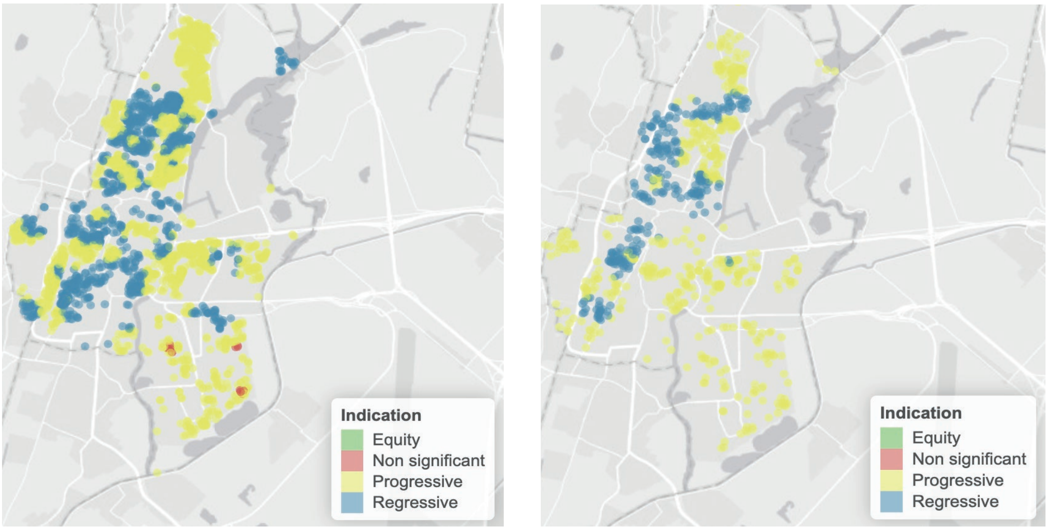


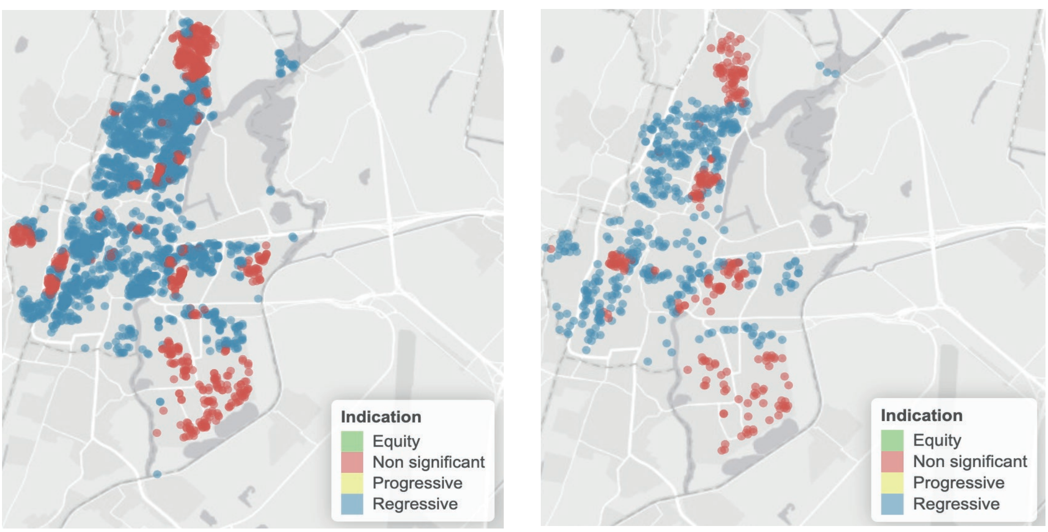
FIGURE 4 ▸ SPATIALLY EXTENDED KOCHIN PARKS MEASURE RESULTS



resemble the results for the spatially extended Paglin-Fogarty measure. Specifically, the findings suggest that the majority of the regression outcomes display the presence of regressive vertical inequity. Large pockets of non-significant findings evident in the north and south of the municipality and enclaves within the central area of the municipality.

The spatially extended Kochin-Parks vertical equity measure findings can be observed in Figure 4. Interestingly, unlike the other spatially extended traditional regression-based vertical equity measures, the Kochin-Parks approach clearly exhibits a dominance of progressive vertical inequity – a finding is in line with the global result of the traditional Kochin-Parks measure. Interestingly, this spatially extended Kochin-Parks does also indicate areas of regressive vertical

FIGURE 5 ▸ SPATIALLY EXTENDED BELL MEASURE RESULTS



inequity – notably in the centre of the municipality. The regions which showed to be non-significant in other spatially extended measures are predominantly progressive in this approach.

For the Bell extended vertical equity measure, the majority of the regression outcomes indicate the presence of regressive vertical inequity, a spatial pattern can be identified between the training and test analysis. Both the north end and the south end of the municipality of Haarlem show non-significant returns for the spatially extended vertical equity measure. Further, the results illustrate pockets of non-significance around the periphery of the central area.

Discussion

The results of this study reveal that the choice of market indicator between sale price and assessed value is still relevant and stays present in the spatial extension of the traditional regression-based vertical equity models. Furthermore, the spatial models reveal less non-significant returns if the spatially extended model accounts for non-linearity. Sale prices are often the preferred indicator of market value, since they emerge directly from the market. However, if sale prices are used, it is highly important that the observed sale prices have undergone scrutiny to verify if the sale price captures true market value or if any outside factor affected the sale. The availability and transparency of these sale prices can differ significantly between different legal systems and assessment practices. In some cases, it can therefore be beneficial to use the assessed value as the market indicator. Akin to using sale prices, if the market indicator of choice is the assessed value, it is essential that the assessed values are interpreted to reflect the market value. This typically means that assessments have to be time adjusted in jurisdictions where assessment cycles are not recurrent or infrequent. Furthermore, if the assessed value is used, it is necessary that certain capping procedures are adjusted for before applying the vertical equity models.

Modern regression-based vertical equity models incorporate both the sale price and the assessed value in their model structures. Either through a two-stage regression model as proposed by Clapp (1990) or through the use of a market value proxy such as the PRB (McMillen & Singh 2023). Though despite the emergence of newer modern vertical equity models, academic consensus suggests that the use of multiple models to identify possible problematic vertical inequity is best practice (Carter 2016). Although, some prefer other vertical equity models over traditional regression-based vertical equity models (McMillen and Singh) 2023).

That said, spatially extended regression-based vertical equity measures have the advantage of producing equity maps to identify spatial variation. These maps visualize the different regions within the jurisdiction which may warrant further investigation into the presence of possible vertical inequity. While the spatially extended traditional regression-based vertical equity measures have shown to provide useful insights, they are rather complex. Compared to their non-spatial counterparts, the spatial extension models need to consider the coordinate reference system of the spatial data. Moreover, their configuration also necessitates choices regarding their bandwidths, kernels, or the k-nearest neighbours, a topic of debate and further research endeavours. Alternatively, it is pertinent that vertical equity analysis based on spatial patterns should receive continued academic interest since it could provide a rather intuitive means to provide useful spatial insights into possible unacceptable vertical inequity assessment.

Conclusion

Academic and practical consensus is very much of the view that vertical equity analyses should be conducted through the application of multiple vertical equity measures. This paper argues that this approach should also be layered, firstly via intuitive and visual analysis, for identifying unacceptable vertical inequity. Second, for further in-depth analysis where complex vertical equity measures can be deployed to identify areas of vertical inequity. Spatially extended traditional vertical equity measures as presented in this study certainly provide insights into the latter category, since they require spatial data skills and are rather complex to use. Though complex, these models offer a new way within the vertical equity measurement toolbox.

The exploratory nature of the spatially extended vertical equity measurements requires caution and to an extent expertise for the interpretation of the results. As with any spatial measurement, the spatial composition of the observations and the spatial calibration of the models always influence the outcomes, and therefore the conclusions that can be drawn. Spatially extended vertical equity measures should be used to identify “areas of interest” within a study region for practical assessment office targeting, and in that sense be complimentary to existing methods. They are not a panacea for solving the challenge of inequitable valuation assessment. In that regard, they should not be used to identify areas with unacceptable vertical inequity on their own. Finally, it is necessary to state that this paper specifically investigated the use of spatially extended regression-based vertical equity measures on single-family residential real estate. The application of the same framework on multifamily residential real estate should be considered for further research, especially in the context of k-nearest neighbours selection methods.

Finally, vertical equity is receiving increased attention worldwide, both in academia and in industry. Given current developments in the Dutch tax system, equity in assessed values may perhaps be of increased importance in the near future. The quality of assessments will most likely receive even more attention on the national level. It is therefore highly important that vertical equity measures are developed and researched within the Dutch context and legislation. This paper pioneers this avenue for research within the Dutch context and calls for both overview studies of existing measures and the development of new measures within the Dutch context.

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FOOTNOTE

1 The broader vertical equity debate is well covered in the literature, and readers are referred to Sirmans et al. (2008), Carter (2016), McMillen & Singh (2023), and Quintos et al. (2023) for detailed discussions.

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